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L^1-convergence of complex double Fourier series

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Abstract. It is proved that the complex double Fourier series of an integrable function $f(x,y)$ with coefficients $\{c_{jk}\}$ satisfying certain conditions, will converge in L^1-norm. The conditions used here are the combinations of Tauberian condition of Hardy–Karamata kind and its limiting case. This paper extends the result of Bray [1] to complex double Fourier series.

Keywords. Complex double Fourier series; L^1-convergence of Fourier series; Cesàro means; de Ia Vallée-Poussin means; Tauberian condition of Hardy–Karamata kind.

1. Introduction

Let $\{c_{jk}\}$ be the Fourier coefficients of an integrable function $f(x,y)$ on the two-dimensional torus $T^2 = [-\pi,\pi)\times[-\pi,\pi)$.

We consider the double Fourier series of f,

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_{jk}\mathrm{e}^{i(jx+ky)} \tag{1.1}$$

with rectangular partial sums $S_{mn}(x,y)$ and Cesàro means $\sigma_{mn}(x,y)$ defined by

$$S_{mn}(f;x,y) = \sum_{|j|\le m}\sum_{|k|\le n} c_{jk}\mathrm{e}^{i(jx+ky)},$$

$$\sigma_{mn}(f;x,y) = \frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n} S_{jk}(x,y) \qquad (m,n\ge 0)$$

and for $\lambda > 1$, the de Ia Vallée-Poussin means by

$$V_{mn}^{\lambda}(f,x,y) = \frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} S_{jk}(x,y),$$

where $[t]$ stands for the greatest integer $\le t$, for a positive real number t.

For f in $L^1(T^2)$, $\|f\| = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x,y)|\mathrm{d}x\,\mathrm{d}y$ stands for the L^1-norm.

The aim of this paper is to show that for a function f in $L^1(T^2)$, under certain conditions on the double Fourier series, the rectangular partial sums S_{mn} converge in L^1-norm to f, that is, $\|S_{mn}-f\|\to 0$ as $m,n\to\infty$. Our result belongs to the extensive study in the literature of what are called L^1-convergence classes in one as well as two variables. In the

single variable case, we particularly refer to [1], [2], [3] (see [2] for further references) and for double Fourier series we are aware of [4] and [8]. For functions f of a single variable with the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, a set of conditions on the coefficients c_n is said to define an L^1-convergence class if, under these conditions,

$$\|S_n - f\| = o(n) \text{ if and only if } c_n \log n = o(n). \tag{1.2}$$

Here S_n is the nth partial sum of the Fourier series, $S_n = \sum_{k=-n}^{n} c_k e^{ikx}$ and $\|\cdot\|$ is the L^1-norm on $[-\pi,\pi)$. The obvious modification in the definition of L^1-convergence class for double Fourier series is to demand instead of (1.2)

$$\|S_{mn} - f\| \to 0 \text{ if and only if } c_{mn} \log m \log n \to 0 \text{ as } \min\,(m,n) \to \infty.$$

Our result is weaker in not having any "only if" part as in the above. To put the conditions in our result in perspective we quote the condition for single Fourier series

$$\lim_{\lambda\downarrow 1} \overline{\lim_{n\to\infty}} \sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta c_k|^p = 0, \qquad (1 < p \le 2). \tag{1.3}$$

This condition used in [1] and [2] is a Tauberian condition of the Hardy–Karamata kind [3] and is a weaker version of the conditions used by Fomin [5], Kolmogorov [6], Littlewood [7] and Teljakovskii [9]. A limiting condition of (1.3) as $p \to 1$ is

$$\lim_{\lambda\downarrow 1} \overline{\lim_{n\to\infty}} (\log n) \sum_{|k|=n}^{[\lambda n]} |\Delta c_k| = 0 \tag{1.4}$$

which can be thought of as the prototype of the conditions we use in our result (conditions (3.1) to (3.4) below). Our conditions have points of contact with [8] and our results are simpler than those of [8] and [9]. Our methods are heavily dependent on some identities in trigonometric series involving the de la Valleé-Poussin means V_{mn}^{λ}. The identities are developed in §2 and the main result is proved in §3.

We end this section by explaining some notations.

Let

$$\begin{aligned} &\triangle_{00} c_{jk} = c_{jk}; \\ &\triangle_{pq} c_{jk} = \triangle_{p-1,q} c_{jk} - \triangle_{p-1,q} c_{j+1,k} \qquad (p \ge 1), \\ &\triangle_{pq} c_{jk} = \triangle_{p,q-1} c_{jk} - \triangle_{p,q-1} c_{j,k+1} \qquad (q \ge 1). \end{aligned}$$

We mention that a double induction argument gives

$$\triangle_{pq} c_{jk} = \sum_{s=0}^{p} \sum_{t=0}^{q} (-1)^{s+t} \binom{p}{s} \binom{q}{t} c_{j+s,\,k+t}.$$

In this paper, we use the differences $\triangle_{pq}$ with $p, q \le 2$.

For $f \in L^1(T^2)$, the symbols $S_{mn}(f)$ and $S_{mn}(f,x,y)$ will have the same meaning as S_{mn}.

Similarly

$$\sigma_{mn}(f) = \sigma_{mn}(f;x,y) = \sigma_{mn}$$

and

$$V_{mn}^{\lambda}(f) = V_{mn}^{\lambda}(f;x,y) = V_{mn}^{\lambda}.$$

In the sequel $\lambda_n = [\lambda n]$ where n is a positive integer and $\lambda > 1$ is a real number.

Finally, we define the functions $E_{0+}(x) = E_{0-}(x) = \frac{1}{2}, E_n(x) = \sum_{k=0}^{n} \mathrm{e}^{ikt}, n \in \mathbb{Z}$ and $E_{-n}(x) = E_n(-x)$. We have $\|E_k(x)\| \le C \log|k|$, where C is an absolute constant.

2. Lemmas

In order to establish our result, we need the following lemmas:

Lemma 2.1[2]. *For $m,n \ge 0$ and $\lambda > 1$, the following representation holds*:

$$\begin{aligned} S_{mn} - \sigma_{mn} = {} & \frac{\lambda_m+1}{\lambda_m-m}\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{\lambda_m,\,\lambda_n} - \sigma_{\lambda_m,\,n} - \sigma_{m,\,\lambda_n} + \sigma_{mn}) \\ & + \frac{\lambda_m+1}{\lambda_m-m}(\sigma_{\lambda_m,\,n} - \sigma_{mn}) + \frac{\lambda_n+1}{\lambda_n-n}(\sigma_{m,\,\lambda_n} - \sigma_{mn}) \\ & - \sum_{|j|\le m} \sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk}\mathrm{e}^{i(jx+ky)} \\ & - \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|\le n} \frac{\lambda_m+1-|j|}{\lambda_m-m} c_{jk}\mathrm{e}^{i(jx+ky)} \\ & - \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_m+1-|j|}{\lambda_m-m}\frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk}\mathrm{e}^{i(jx+ky)}. \end{aligned}$$

Lemma 2.2. *For $m,n \ge 0$ and $\lambda > 1$, we have the following representation*:

$$\begin{aligned} V_{mn}^{\lambda} - S_{mn} = {} & \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_m+1-|j|}{\lambda_m-m}\frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk}\mathrm{e}^{i(jx+ky)} \\ & + \sum_{|j|\le m} \sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk}\mathrm{e}^{i(jx+ky)} \\ & + \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|\le n} \frac{\lambda_m+1-|j|}{\lambda_m-m} c_{jk}\mathrm{e}^{i(jx+ky)}. \end{aligned}$$

Proof. We have

$$V_{mn}^{\lambda}(f,x,y) = \frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} S_{jk}(x,y).$$

Performing the double summation by parts, we have

$$\begin{aligned}
V_{mn}^{\lambda} &= \frac{\lambda_m+1}{\lambda_m-m}\frac{\lambda_n+1}{\lambda_n-n}\sigma_{\lambda_m,\,\lambda_n} - \frac{\lambda_m+1}{\lambda_m-m}\frac{n+1}{\lambda_n-n}\sigma_{\lambda_m,\,n} \\
&\quad - \frac{m+1}{\lambda_m-m}\frac{\lambda_n+1}{\lambda_n-n}\sigma_{m,\,\lambda_n} + \frac{m+1}{\lambda_m-m}\frac{n+1}{\lambda_n-n}\sigma_{mn} \\
&= \frac{\lambda_m+1}{\lambda_m-m}\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{\lambda_m,\,\lambda_n} - \sigma_{\lambda_m,\,n} - \sigma_{m,\,\lambda_n} + \sigma_{mn}) \\
&\quad + \frac{\lambda_m+1}{\lambda_m-m}(\sigma_{\lambda_m,\,n} - \sigma_{mn}) + \frac{\lambda_n+1}{\lambda_n-n}(\sigma_{\lambda_m,\,n} - \sigma_{mn}) + \sigma_{mn}
\end{aligned}$$

The use of Lemma 2.1, gives

$$\begin{aligned}
V_{mn}^{\lambda} - S_{mn} &= \sum_{|j|=m+1}^{\lambda_m}\sum_{|k|=n+1}^{\lambda_n}\frac{\lambda_m+1-|j|}{\lambda_m-m}\frac{\lambda_n+1-|k|}{\lambda_n-n}c_{jk}\mathrm{e}^{i(jx+ky)} \\
&\quad + \sum_{|j|\le m}\sum_{|k|=n+1}^{\lambda_n}\frac{\lambda_n+1-|k|}{\lambda_n-n}c_{jk}\mathrm{e}^{i(jx+ky)} \\
&\quad - \sum_{|j|=m+1}^{\lambda_m}\sum_{|k|\le n}\frac{\lambda_m+1-|j|}{\lambda_m-m}c_{jk}\mathrm{e}^{i(jx+ky)}.
\end{aligned}$$

Lemma 2.3. *For $m,n \ge 0$ and $\lambda > 1$, we have*

$$\begin{aligned}
&\sum_{j\le m}\sum_{|k|=n+1}^{\lambda_n}\frac{\lambda_n+1-|k|}{\lambda_n-n}c_{jk}\mathrm{e}^{i(jx+ky)} \\
&= \sum_{j=0\pm}^{m-1}\sum_{|k|=n}^{\lambda_n-1}\frac{\lambda_n-|k|}{\lambda_n-n}\Delta_{11}c_{jk}E_j(x)E_k(y) \\
&\quad + \frac{1}{\lambda_n-n}\sum_{|j|=\pm 0}^{m-1}\sum_{|k|=n+1}^{\lambda_n}\Delta_{10}c_{jk}E_j(x)E_k(y) \\
&\quad - \sum_{|j|=0\pm}^{\lambda_n}\sum_{k=n}\Delta_{10}c_{jk}E_j(x)E_k(y) \\
&\quad + \sum_{j=m}\sum_{|k|=n}^{\lambda_n-1}\frac{\lambda_n-|k|}{\lambda_n-n}\Delta_{01}c_{jk}E_j(x)E_k(y) \\
&\quad + \frac{1}{\lambda_n-n}\sum_{|j|=m}\sum_{|k|=n+1}^{\lambda_n}c_{jk}E_j(x)E_k(y) \\
&\quad - \sum_{|j|=m}\sum_{|k|=n}c_{jk}E_j(x)E_k(y).
\end{aligned}$$

Proof. By summation by parts,

$$\sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk} e^{iky}$$

$$= \sum_{k=n}^{\lambda_{n-1}} \Delta_{01}\left[\frac{\lambda_n+1-k}{\lambda_n-n} c_{jk}\right] E_k(y) + \frac{1}{\lambda_n-n} c_{j,\,\lambda_n} E_{\lambda_n}(y)$$

$$- \frac{\lambda_n-n+1}{\lambda_n-n} c_{j,\,n} E_n(y)$$

$$= \sum_{k=n}^{\lambda_{n-1}} \frac{\lambda_n-k}{\lambda_n-n} \Delta_{01} c_{jk} E_k(y) + \frac{1}{\lambda_n-n} \sum_{k=n+1}^{\lambda_n} c_{jk} E_k(y) - c_{j,\,n} E_n(y).$$

Similarly

$$\sum_{k=n+1}^{\lambda_n} \frac{\lambda_n+1-k}{\lambda_n-n} c_{j,-k} e^{-iky}$$

$$= \sum_{k=n}^{\lambda_{n-1}} \frac{\lambda_n-k}{\lambda_n-n} \Delta_{01} c_{j,\,-k} E_{-k}(y) + \frac{1}{\lambda_n-n} \sum_{k=n+1}^{\lambda_n} c_{j,\,-k} E_{-k}(y)$$

$$- c_{j,\,-n} E_{-n}(y).$$

Combining the above results

$$\sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk} e^{iky}$$

$$= \sum_{|k|=n}^{\lambda_n-1} \frac{\lambda_n-|k|}{\lambda_n-n} \Delta_{01} c_{jk} E_k(y) + \frac{1}{\lambda_n-n} \sum_{|k|=n+1}^{\lambda_n} c_{jk} E_k(y)$$

$$- \sum_{k=|n|} c_{jk} E_k(y).$$

Another summation by parts with respect to j gives the required result.

Lemma 2.4. *For $m,n \geq 0$ and $\lambda > 1$, we have*

$$\sum_{|j|=m+1}^{\lambda_m} \sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_m+1-|j|}{\lambda_m-m} \frac{\lambda_n+1-|k|}{\lambda_n-n} c_{jk} e^{i(jx+ky)}$$

$$= \sum_{|j|=m}^{\lambda_m-1} \sum_{|k|=n}^{\lambda_n-1} \frac{\lambda_m-|j|}{\lambda_m-m} \frac{\lambda_n-|k|}{\lambda_n-n} \Delta_{11} c_{jk} E_k(x) E_k(y)$$

$$+ \frac{1}{\lambda_n-n} \sum_{|j|=m}^{\lambda_{m-1}} \sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_m-|j|}{\lambda_m-m} \Delta_{10} c_{jk} E_j(x) E_k(y)$$

$$+ \frac{1}{\lambda_m-m} \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|=n}^{\lambda_{n-1}} \frac{\lambda_n-|k|}{\lambda_m-m} \Delta_{01} c_{jk} E_j(x) E_k(y)$$

$$
\begin{aligned}
&- \sum_{|j|=m}^{\lambda_m - 1} \sum_{|k|=n} \frac{\lambda_m - |j|}{\lambda_m - m} \Delta_{10} c_{jk} E_j(x) E_k(y) \\
&- \sum_{j=m} \sum_{k=n}^{\lambda_n - 1} \frac{\lambda_m - |k|}{\lambda_n - n} \Delta_{01} c_{jk} E_j(x) E_k(y) \\
&- \frac{1}{\lambda_m - m} \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|=n} c_{jk} E_j(x) E_k(y) \\
&- \frac{1}{\lambda_n - n} \sum_{|j|=m} \sum_{|k|=n+1}^{\lambda_n} c_{jk} E_j(x) E_k(y) \\
&+ \frac{1}{\lambda_n - n} \frac{1}{\lambda_m - m} \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|=n+1}^{\lambda_n} c_{jk} E_j(x) E_k(y) \\
&- \sum_{|j|=m} \sum_{|k|=n} c_{jk} E_j(x) E_k(y).
\end{aligned}
$$

Proof of this lemma follows on similar lines as in Lemma 2.3.

3. Main result

The main result of this paper is the following theorem:

Theorem 3.1. *Let* $f \in L^1(T^2)$*, and* $\{c_{jk}\}$ *be its Fourier coefficients. If*

$$\sum_{|j|=0\pm}^{\infty} (\log|j|)(\log|k|)|\Delta_{10} c_{jk}| \to 0, \ \text{as} \quad |k| \to \infty, \tag{3.1}$$

$$\sum_{|k|=0\pm}^{\infty} (\log|j|)(\log|k|)|\Delta_{01} c_{jk}| \to 0, \ \text{as} \quad |j| \to \infty, \tag{3.2}$$

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \sum_{|j|=0\pm}^{\infty} \sum_{|k|=n}^{\lambda_n} (\log|j|)(\log|k|)|\Delta_{11} c_{jk}| = 0, \tag{3.3}$$

$$\lim_{\lambda \downarrow 1} \overline{\lim_{m \to \infty}} \sum_{|k|=0\pm}^{\infty} \sum_{|j|=m}^{\lambda_m} (\log|j|)(\log|k|)|\Delta_{11} c_{jk}| = 0. \tag{3.4}$$

Then $\|S_{mn}(f) - f\| = o(1)$ *as* $\min(m,n) \to \infty$.

Proof of the Theorem. Since $f \in L^1(T^2)$ therefore $\|\sigma_{mn}(f) - f\| = o(1)$ as $\min(m,n) \to \infty$ (see, e.g. [10], vol. 2, p. 309). It follows that $\|V_{mn}^{\lambda} - f\| = o(1)$ as $(m,n) \to \infty$.

Consequently it is sufficient to prove that

$$\|V_{mn}^{\lambda} - S_{mn}\| = o(1) \ \text{ as } \ \min(m,n) \to \infty.$$

Combining the results of Lemmas 2.1–2.4, we have

$$\begin{aligned}
V_{mn}^{\lambda}-S_{mn} &= R_1^{\lambda}(m,n;x,y)+R_2^{\lambda}(m,n;x,y)-R_3^{\lambda}(m,n;x,y)\\
&\quad -R_4^{\lambda}(m,n;x,y)+R_5^{\lambda}(m,n;x,y)-R_0^{\lambda}(m,n;x,y),\\
R_1^{\lambda}(m,n;x,y) &= \sum_{|j|=m}^{\lambda_m-1}\sum_{|k|=n}^{\lambda_n-1}\frac{\lambda_m-|j|}{\lambda_m-m}\frac{\lambda_n-|k|}{\lambda_n-n}\Delta_{11}c_{jk}E_j(x)E_k(y)\\
&\quad +\sum_{|j|=0\pm}^{m-1}\sum_{|k|=n}^{\lambda_n-1}\frac{\lambda_n-|k|}{\lambda_n-n}\Delta_{11}c_{jk}E_j(x)E_k(y)\\
&\quad +\sum_{|j|=m}^{\lambda_m-1}\sum_{|k|=0\pm}^{n-1}\frac{\lambda_m-|j|}{\lambda_m-m}\Delta_{11}c_{jk}E_j(x)E_k(y),\\
R_2^{\lambda}(m,n;x,y) &= \frac{1}{\lambda_n-n}\sum_{|j|=0\pm}^{m-1}\sum_{|k|=n+1}^{\lambda_n-1}\Delta_{10}c_{jk}E_j(x)E_k(y)\\
&\quad +\frac{1}{\lambda_m-m}\sum_{|j|=m+1}^{\lambda_m}\sum_{|k|=0\pm}^{n-1}\Delta_{01}c_{jk}E_j(x)E_k(y)\\
&\quad +\frac{1}{\lambda_n-n}\sum_{|j|=m+1}^{\lambda_m}\sum_{|k|=n}^{\lambda_n-1}\frac{\lambda_m-|j|}{\lambda_m-m}\Delta_{10}c_{jk}E_j(x)E_k(y)\\
&\quad +\frac{1}{\lambda_m-m}\sum_{|j|=m+1}^{\lambda_m}\sum_{|k|=n}^{\lambda_n-1}\frac{\lambda_n-|j|}{\lambda_n-n}\Delta_{01}c_{jk}E_j(x)E_k(y),\\
R_3^{\lambda}(m,n;x,y) &= \sum_{|k|=n}\sum_{|j|=0\pm}^{m-1}\Delta_{10}c_{jk}E_j(x)E_k(y),\\
R_4^{\lambda}(m,n;x,y) &= \sum_{|j|=m}\sum_{|k|=0\pm}^{n-1}\Delta_{01}c_{jk}E_j(x)E_k(y),\\
R_5^{\lambda}(m,n;x,y) &= \frac{1}{\lambda_m-m}\frac{1}{\lambda_n-n}\sum_{|j|=m+1}^{\lambda_m}\sum_{|k|=n+1}^{\lambda_n}c_{jk}E_j(x)E_k(y),\\
R_0^{\lambda}(m,n;x,y) &= \sum_{|j|=m}\sum_{|k|=n}c_{jk}E_j(x)E_k(y).
\end{aligned}$$

It follows from (1.6),

$$\begin{aligned}
\|R_1^{\lambda}(m,n;x,y)\|_1 \le C\Bigg\{&\sum_{|j|=0\pm}^{\infty}\sum_{|k|=n}^{\lambda_n}(\log|j|)(\log|k|)|\Delta_{11}c_{jk}|\\
&+\sum_{|j|=0\pm}^{\infty}\sum_{|k|=n}^{\lambda_n}(\log|j|)(\log|k|)|\Delta_{11}c_{jk}|\Bigg\}.
\end{aligned}$$

Thus, by (3.3) and (3.4), we get

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\|R_1^\lambda(m,n;x,y)\|_1 = 0.$$

$R_2^\lambda(m,n;x,y) = R_{21}^\lambda(m,n;x,y) + R_{22}^\lambda(m,n;x,y)$, where

$$\begin{aligned}
R_{21}^\lambda(m,n;x,y) &= \frac{1}{\lambda_n - n}\sum_{|j|=0\pm}^{m-1}\ \sum_{|k|=n+1}^{\lambda_n}\Delta_{10}c_{jk}E_j(x)E_k(y)\\
&\quad + \frac{1}{\lambda_n - n}\sum_{j=m}^{\lambda_m-1}\ \sum_{|k|=n+1}^{\lambda_n}\frac{\lambda_m-|j|}{\lambda_m-m}\Delta_{10}c_{jk}E_j(x)E_k(y),\\
R_{22}^\lambda(m,n;x,y) &= \frac{1}{\lambda_m - m}\sum_{|j|=m+1}^{\lambda_m}\ \sum_{|k|=\pm 0}^{n-1}\Delta_{01}c_{jk}E_j(x)E_k(y)\\
&\quad + \frac{1}{\lambda_m - m}\sum_{|j|=m+1}^{\lambda_m}\ \sum_{|k|=n}^{\lambda_n-1}\frac{\lambda_n-|k|}{\lambda_n-n}\Delta_{01}c_{jk}E_j(x)E_k(y),\\
R_{21}^\lambda(m,n;x,y) &= \frac{1}{\lambda_n - n}\sum_{|j|=0}^{m-1}\ \sum_{|k|=n+1}^{\lambda_n}\Delta_{10}c_{jk}E_j(x)E_k(y)\\
&\quad + \frac{1}{\lambda_n - n}\sum_{|j|=m}^{\lambda_{m-1}}\ \sum_{|k|=n+1}^{\lambda_n}\frac{\lambda_m-|j|}{\lambda_m-m}\Delta_{10}c_{jk}E_j(x)E_k(y)\\
&= \frac{1}{(\lambda_m-m)(\lambda_n-n)}\\
&\quad \times\sum_{|i|=m}^{\lambda_{m-1}}\ \sum_{|k|=n+1}^{\lambda_n}\left(\sum_{|j|=0}^{i}\Delta_{10}c_{jk}E_j(x)E_k(y)\right).
\end{aligned}$$

By (1.6) and (3.1), for $\lambda > 1$, we conclude that

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\|R_{21}^\lambda(m,n;x,y)\|_1 = 0.$$

Similarly, by (1.6) and (3.2), for $\lambda > 1$, we have

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\|R_{22}^\lambda(m,n;x,y)\|_1 = 0.$$

By (3.1) and (3.2), we get

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\|R_3^\lambda(m,n;x,y)\|_1 = 0,$$

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\|R_4^\lambda(m,n;x,y)\|_1 = 0.$$

Taking into account $|c_{jk}| \le \sum_{|u|=j}^{\lambda_n}|\Delta_{10}c_{uk}|$, and by (3.1), we find that $c_{jk}\log|j|\log|k| = o(1)$ as $\min(|j|,|k|)\to\infty$.

Combining this with (1.6), we conclude that

$$|R_5^\lambda(m,n;x,y)| \le \max_{\substack{m<u\le\lambda_m \\ n<v\le\lambda_n}} \|R_0^\lambda(u,v;x,y)\| \to \infty \text{ as } \min(m,n)\to\infty.$$

Combining all what we have done so far, we conclude the desired result.

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